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Mind Pollution

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Introduction. The accelerating deterioration of our environments due to our activities is now attracting more and more attention. Waste products pollute the air, the waters, the land, and even the moon. Overpopulation is another polluting threat. Yet the primary attention given to these polluting activities is directed to the more or less immediate causes rather than primary causes. One primary cause of the irresponsibility we demonstrate lies in the establishments of attitudes, habits, goals, and philosophies. This may be called mind pollution. In the U.S.A., for example, much has been said about the detrimental effects of cigarette smoking. Yet despite the mild measures against the spread of this habit, the tobacco companies have managed by innuendo and spurious advertising to boost their sales. Every cigarette vending machine accessible to the public is a de-facto violation of the laws against selling cigarettes to minors. Yet we condone this illegal activity. Why?

There are many obvious manifestations of attitudes which contribute to the pollution of the environment. For example, in the 1960 presidential election in the U.S.A. one prominent issue was increase of the gross national product. As an objective increasing the gross national product is presumably reasonable for developing countries, but in the U.S.A. it must be considered unethical and even immoral. The profit motive has now been completely oversold. The number of associations, unions, and corporate units which accept this

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motive as a primary one has increased alarmingly. Each measures success by money rather than by contribution to the common good. A man who accumlates great wealth but pays no taxes and effectively contributes nothing is an object of envy rather than scorn. An excellent teacher who contributes much but is paid little is an object of neglect. Why? If the values prized in a nation are those which are against its population's interests, it is only a matter of time before it will reap the harvest it deserves.

Now it might be thought that there are many areas of human endeavor and activity which provide models of improved behavior. Yet, they are comparatively few in number, even those with explicitly advertised high ideals falling far short of their presumed goals. Thus the medical and legal professions with their proclaimed high ideals often take concerted action against the public interests. The improvement of health and justice has become, in effect, side shows to direct the public attention from the acceptance by these professions of the profit motive.

Until recent years scientists managed to project a public image of high ideals. It has now become evident that the scientific organizations are also tainted with exaggerated self-interest and that scientists and engineers are often the means by which pollution is promoted. Who, for example, has the right to alter the Van Allen belt by setting off hydrogen bombs? To whom does oil, being consumed so wastefully without replacement really belong?

Religious institutions have failed to demonstrate adequacy for the times. In this case the lust for power has indeed corrupted the clerics, who seemingly sanctify all kinds of injustices if done in the name of God. Of course, we might well agree that there are many guidelines for improvement of behavior in the basic tenets of many religions. However, the miracles produced by scientists and engineers overshadow those produced or claimed by religions. Psychiatrists try to cast out devils and collect more for the attempts than most ministers and priests. The spectacles of sports arouse more enthusiasm than religious organizations seem able to. This may be because the sacrifices required of participants are obvious and because the spectators assume a mood of participation and partisanship.

Universities and institutions of higher learning have been assumed to represent the best in thinking and teaching. Yet the goals for these have led to extremes of specialization and to disregard the process of educating. The improvement of education in recent years has been almost negligible compared to the opportunities. In fact, many recently established practices seem detrimental to proper education.

In this essay, I concentrate in one important aspect of mind pollution, that of education in mathematics. Now mathematics, in contrast to many other aspects of knowing, is basically simple having been and being created by people. What I mean by saying mathematics is simple is that there is no area of mathematics of which I am aware which could not be learned during or before the secondary school level, if it were conceded to be important enough. Other areas such as sociology, history, political science, biology, psychology, law, and medicine have aspects for which adulthood may be more important. Mathematics is taught to every school child over a number of years. Only the study of language competes with mathematics for the time it uses in schools. For this reason it is important that education in mathematics be optimal and effective.

I find that the neglect of mathematics education is closely related to the attitudes of professional mathematicians. I demonstrate here several instances of the failure of mathematicians to properly interpret the most important concepts of mathematics. I will also indicate some of the instances of abuses of terminology. My interpretation of the actions needed to improve mathematics education cannot be said to have been tested or even considered by mathematics educators. The basic problem is to establish a cybernetic system which enables continuing progress in education on all levels. No rigid system of schooling can be adjustable enough to provide adequate educational opportunities.

Failure of Mathematics Education

A basic goal of mathematics education is to assure a level of understanding of mathematics--what it is, what it does, and what it fails now to do. Ask any baccalaureate degree holder with a major in mathematics what mathematics is. He will not be able to give a sensible answer, no matter how much time he is given. Ask a Ph.D. in mathematics the same question. He also will not know, or else he may reply with some nonsense statements like "mathematics is deductive science" or "mathematics is what mathematicians do."

Is it reasonable that not even a Ph.D. in mathematics will know what a theorem or a proof is when every school teacher should know? I say it is not reasonable! It might be said that mathematics is too difficult to be presented in a meaningful way. I claim that the emphasis on technical results has led to the anomaly that the understanding of mathematics has been left out of consideration.

Chemistry and biology are inherently at least as difficult as mathematics. Yet, a one year course in either chemistry or biology will give the student a better cultural view of these subjects than he can achieve now after years of study of mathematics. It is possible to do much better with mathematics.

What is wrong with the present education in mathematics? The general ideas which relate mathematics to other human activities not only are not taught, they are not known! There are no effective presentations of mathematics as a whole. Its structure is not outlined. Concepts which could have significance beyond mathematics are trivialized so as to be merely of mathematical interest. Big ideas are reduced to the ashes of axiomatics. General principles and approaches which provide patterns for details are simply omitted.

Functions and Binary Relations.

I choose first to present spectacular instance of the separation of mathematics from reality by the consistent abuse of one of its most important concepts--"function." So important functions are deemed that some mathematicians have recommended that they be made a basis for all mathematics. Various forms of definitions have been proposed no one of which is particularly useful in dealing with functions. Certain textbooks suggest "transformation" and "map" or "mapping" as synonyms for "function." Some years ago I realized that "transformer" and "mapper" are better for the purpose in some contexts.

Suddenly, one evening in March, 1970, after giving a lecture on computer science, it occurred to me that a function corresponds to a <u>verb</u> of a special kind. Thus $y = f(x)$ can be diagrammed as <u>x|f|y</u> or xfy in which x is the nominative, f is the verb, and y is the object, direct or indirect of the verb. The practice of writing a binary relation in the form xRy, "x is related to y" indicates the sentence diagram of binary relations.

In speaking of functions the various statements "x goes into y", "x becomes y", "x determines y", "x is labelled y", and "x is represented by y" shows that the function is usually considered as a verb in the present tense.

I have tried this interpretation of function on a number of mathematicians. They have, without exception, agreed that it is superior, conceptually, to the previous interpretation. Now I ask: How have people for such a long period of time (nearly 200 years) who "knew" what verbs and functions are, not recognize this now obvious comparison? It appears that we have been <u>mesmerized</u> into accepting the separation of mathematical concepts from those in other areas, in particular, language. We have been inhibited from recognizing patterns which should have been obvious, mathematics is split from interaction with other areas.

Since this example is so illuminating, I pursue it further. The present tense ordinarily used suggested using other sentence forms. Consider then "x will go into y (at a later time)", "x went into y", "x will have determined y", "x might be represented by y", "x should go into y", "x ought to become y", "probably x will go into y", and you will have a variety of ways of thinking of functions not all of which have mathematical models now. The imperative and conditional imperative modes of sentences correspond to the controlling devices of computing. A <u>control</u> function is one which tells other functions when to act and on what. The program formats DO, LET, GO TO, IF---THEN show the imperative and conditional imperative aspects of the control of computers.

The capability of computers to "learn" is based in implementation of the IF---THEN commands. I believe that an effective theory of computers can now be based on functions and relations, the relations being the state of the machine at a given cycle time and the functions carrying one relation into the next relation or state.

A computer, incidentally is a transformer, not a transformation; a mapper, not a mapping. The confusion of the object of the sentence with the verb is, most regretably, standard practice. A university is a (partial) transformer of students, it is not a transformation.

The lesson to be drawn from the above example is obvious. A most important concept of mathematics has never been well-treated. How many more such fiascoes are there? I can mention a few here. Functions which associate sets with sets are everywhere in mathematics. The use of function notation would greatly simplify many of the operations now achieved clumsily, but there is a strange taboo against using the notation. In topology, for example, closure, interior, boundary, and complement indicate set-valued set-functions. Moreover, the function which associates with each point the set of all its neighborhoods has values which are collections of sets!

In concluding this section let me take another example from binary relations. Transitive binary relations are very special kinds of binary relations important in all of mathematics. That they are rather rare on readily be seen by considering graphs of functions or by considering almost any planar figure as a binary relation in the set of real numbers. Yet there is a superstition, which permeates the mathematical texts and treatise that transitive relations are _too general_ for study. Worse, it has been stated that lattices, which are special kinds of reflexive antisymmetric transitive relations, are almost too general to be considered mathematical objects. This comprises mind pollution!

In the common languages, comparative adjectives applied appropriately indicate transitive relations which are usually not reflexive and not antisymmetric. Yet they yield order relations. The terminology of order relations would do credit to a cabalistic society but is unfit for education. This is another phase of mathematics education which could be grossly improved. Terminology should not consistently be against reason. For example, a strict order relation _must_ be an order relation if we are to use language properly. Yet it is not in the treatises and texts of mathematics in general. In many uses it appears as if the terminology were deliberately chosen to be confusing. If making mathematics as clear as we can shows it to be trivial, which I doubt, then it is trivial. Only by clearing away the misinterpretations by which issues have been befogged can we glimpse the deeper mysteries beneath.

Filters and neighborhoods.

For years I have struggled with the concept of "neighborhood" trying to find out what it means basically--i.e. to the non-topologists. Finally I arrived at the following interpretation. The neighborhoods of an objective are the conditions which _must_ be met to achieve the objective. Thus neighborhoods _protect_ an objective from trivial attainment. For example, to achieve a B.S. degree in a university the student must meet stated conditions which are the neighborhoods of the degree. If he meets all conditions he has converged i.e. he gets the degree. This conceptualization of "neighborhood" is simple, it can be explained to children and it relates the topological concept to a much wider range of human activities. Now, in the present state of education you will never see such a simple and non-technical discussion of neighborhood. If topology is made trivial by such interpretations, topology is trivial. The filling of minds with technical concepts without establishing its relationship is a form of pollution. Topology would be much more useful if more people understood it in their terms.

I once read a technical definition of "filter" in a topology text. It was not satisfactory to me, so I asked several topologists "how do topology filters filter?" They did not know! I decided then to define filters myself by considering the filters I knew about--pipe filters, chemistry filters, air filters, and electronic filters. I then, in a few minutes, decided that a filter in a set is any device which passes or does not pass each element in the set.

That is to say, a filter is a device which makes binary decisions. Now I emphasized the device since in the examples I had selected the filter was a mechanism for producing the decisions, it was not the result of its application. In a rather short time I had defined relational filters and learned how to interpret the neighborhood filters of topology. The collection of all neighborhoods of a point is called the neighborhood filter of the point in topology. Its filtering action is to pass all sets which are close to the point (the point is in the closure of each accepted set) and to reject all other sets. I ask why do topologists use the term filter and divorce its interpretation from that of other filters? It is against the interest of good education to use terms in this way.

Now filters as I defined them become a unifying concept for mathematics and also have practical examples accessible everywhere. Equations are filters, inequalities are filters, an axiom system is a filter, definitions are filters and so on. I feel most strongly that by early and repeated use of such a general concept, mathematics education can be made more enjoyable, the interrelationships many numbers of concepts and structures previously regarded as unrelated can be shown. Since I first defined my concept of filter in 1967 it is reasonable to ask: How many educational opportunities have been missed? The answer, I believe, is that there are many. Why was the simple and general definition I have produced not given before? I think it is due to the low esteem in which professional mathematicians hold understanding and education.

I cannot go into the numbers of cases in which filters appear and might be used to simplify education. Arithmetic operations can be interpreted as filtering devices. A system of linear equations is a filter which is a conjunction of filters. Solving exactly the system amounts to replacing the original filter by a sequence of equivalent filters. An equation involving square roots is a filter. This filter is often replaced by a coarser filter which accepts "extraneous" roots from which must be passed through the original filter for final acceptance. All through mathematics instances of filters occur. In some places they should be mentioned explicitly, in others perhaps they should not.

My conclusion is that if mathematics is not worth understanding it occupies too much time. Much more work needs to be done on the sense of mathematics.

Information, Approximation, and Continuity

Accessibility and inaccessibility are two of the big concepts relating activities in trade, science, education, governments, religions and what have you. Tradesmen seek access to markets at the same time trying to prevent access to their processes and techniques. Scientists seek access to the mysteries of the universe and so do religions. Education is designed to provide accessibility to certain forms of knowledge--at the same time making other kinds inaccessible. In one interpretation mathematicians increase access to information by providing theorems, language, formulas, methods, and algorithms.

One of the most useful kinds of activity in reducing "real systems" to mathematical systems is variously labelled as abstracting, modelling, or approximating. Thus the plane of geometry may be considered as abstraction or idealization of real surfaces. The advantages of using the geometrical entity are numerous. First, it approximates adequately many surfaces and it is a reasonable replacement. Next, it is simpler than any real surface and it is amendable to manipulation. For its proper uses the plane contains as <u>little</u> information or structure as possible. On the other hand imagining the plane to be comprised of points leads to an enormous number of configurations which had no known counterparts in reality. Some of these configurations then serve as design elements and from these man makes objects to approximate geometrical objects. Accordingly, there are not only mathematical models of real systems, there are also physical models of mathematical "objects,"

It is an unfortunate aspect of mathematics education that many pupils do not well experience this relationship between systems. Yet it is critical that the interactions between mathematics and other activities be clearly understood.

Approximation theorists of modern vintage have confined the term "approximation" to a very small area mainly in linear vector spaces endowed with norms. Yet the idea behind approximation has no need for such an esoteric background. After some years of considering the matter, I have come up with the following approach. The result of an approximating process is the substitution of one entity for another, with the intention that the former shall play the same role in some regards as the latter. For example, oleomargarine is an approximation to butter when it is used as a substitute.

One grievous error in interpreting approximations is to allow only good approximations. In the above example, I may consider oleomargarine as an approximation to butter without making any statement concerning how good it is as an approximation. To some people this approximation is bad, to others it is good (they use oleomargarine), and to others, oleomargarine is superior to butter--they like it better. Who is right?

This same kind of error in making definitions applies to "art", "music", and "creativity" and many other terms. There is no way of defining well any one of these terms if the definitions attempt to weed out bad or poor examples as they always seem to.

Let us now imagine another approximating process. The translating from one language to another of an article is an approximating process. If you say the translation is good, (i.e. it is a good approximation) then you have said that the translating was approximately <u>continuous</u> because it saved the (to you) important features. If someone else says the same translation is bad then he has another criterion. Continuity, I have decided, is dual to invariance. Functions are continuous with respect to whatever they preserve or leave invariant.

Now read one, two or one hundred discussions of approximation and continuity in any mathematics texts of your choice. Will you find any sensible discussion of the concepts? I have yet to see one which does. In education this means that important concepts are being ignored simply because they have been severed from reality and preempted for investigation of their technical and narrow applications.

Certain concepts of mathematics do indeed depend upon a technical background of some depth. But an amazingly large number can be presented on a very low technical level. Unless it is known which concepts can be learned early, how can mathematics education be satisfactory? The fundamentally poor attitude involved is that of disregard for the young and this is a result of mind pollution.

Measures and Distances.

The temptation to lay down axiom systems to define certain mathematical concepts is great. Since 1900 a large number of mathematical systems have been thus formalized. The advantages of axiom systems are well advertised. They provide, in a sense, basic generators or guidelines for the concept defined and thus provide a useful means of developing the underlying assumptions.

The disadvantages of axiomatizations are less well understood. The process of selecting an axiom system is a process of inductive rather than deductive reasoning and it is therefore subject to the level of understanding which the producer of the axiom system possesses. For example, the current definition of a topological space via axioms would scarcely have been accepted if the matter had been given careful thought.

The famous French mathematician Henri Lebesgue developed a generalization of the concepts of arc length, area, and volume. This definition was very well adapted to many problems of analysis. Later studies led to an axiomatic presentation of a natural extension of Lebesgue's definition. Unfortunately, the term "measure" was chosen for this axiomatically defined concept and now several text books have appeared in which this term is used in the sense of the axiomatization.

This use is a clear case of pollution. The so-called measure theorists have a definition of "measure" which they <u>cannot</u> use properly in their own theory! Thus to every literate individual, an external or exterior measure must be a special kind of measure. But no! In measure theory an exterior measure is not usually a measure as defined. Projection measures important in measure theory are also not measures by definition! More important, however, is that the axiomatic definition of measure excluded most of the important and known measures of mathematics. It is clear that the axiomatizers were not well-informed concerning the meanings of measure. For example, take the cardinal number of a set, the dimension of a space, the diameter of a set, the distance between a pair of points, the mean value of a function, the norm of a function or vector, and numbers of others. All these <u>are</u> measures but <u>none</u> falls in the scope of measure theory. In fact, there are reasonable measures which satisfy <u>none</u> of the axioms presumed to define measures.

How much better it would have been if that word had not been so ill treated. In this case, I consider that it is important to not let a small and evidently uninformed sector of mathematicians dictate the use of an important big concept. Is it any wonder that students on whom this usage has been forced cannot grasp mathematics?

Incidentally, I have written up a better definition of measure which does embrace most of the examples I know of. It is at a level which requires no technical background. Since "measure" properly treated is an illuminating and important concept, the carelessness of its treatment in measure theory can only be regarded as reprehensible. I would simply call the special concept "Lebesgue measure", thereby, giving it a much better name.

Now distances are measures of separation or inaccessibility. The first widely accepted axiomatization of distance was given by M. Fréchet as a definition of a metric. Unfortunately this definition again did not embrace the distances known in mathematics at the time it was published. Moreover, there are distances in use in real life which satisfy none of the axioms, for example the net cost of going from one place to another can be negative and is not always symmetric, and satisfies no triangle law. Moreover, there are socially important distances which are not conceptually real-valued. For example, the blood kinship "distance" is not real-valued, identical twins are zero distance apart but are distinct, asymmetry is present, and the analogue of the triangle law fails!

In this case I can see the reasoning behind the definition. The metric system of units is based, in part on the meter, a measure of distance. Moreover, the definition itself stimulated more thinking about geometric type distances then before. The damage here is not due to the term itself, it is that the assumption was made that distances were subsumed in metrics. In most dynamic applications of mathematics, asymmetry is the rule, not the exception. Banach following Fréchet, defined norms to be symmetric, resulting in an inefficient development of approximation theory. Yet Minkowski had earlier proposed asymmetric norms based on convex sets. In my experience, applications of mathematics have been hampered by insistence on symmetric norms. Misdirection due to misinterpretation is quite standard. Fréchet himself, incidentally, also treated asymmetric distances but that work received less notice.

My point here is that virtually no mathematics teacher knows that there are distances all around which are not metrics. If he did know it he could use the facts to excellent advantage.

Ideas Concerning Mathematics Education.

To this point I have discussed a few concepts of recognized importance in mathematics. Let me now consider how the system of education falls short in a general way. The order of difficulty of subjects in mathematics seems to be roughly as follows: algebra, combinatory geometry, arithmetic, infinitesimal geometry including analysis and topology. Actually arithmetic, the way it is taught, may be more difficult than analysis. I rate algebras as least difficult because the axiom systems for algebras are readily expressed in language. This is not to say that there are not unsolved problems and untouched branches of every area. Geometry is difficult because it has many concepts which are basically not verbalizible such as angle, area, curve, plane.

Whether or not arithmetic can be made effectively less difficult, I am not certain. Teaching arithmetic well would seem to require concomitant instruction in the relevant algebras and even at the best, it involves the difficult ideas of rational numbers in which there is available an infinite set of names for each number. The algorithms of arithmetic in themselves comprise a formidable feat of learning.

A child starts off in arithmetic with several functions of two variables (before he has experienced the perhaps less natural functions of one variable.) He is compelled to be a machine, doing things for which computers are much more reliable. Boolean arithmetic is naturally easier and the use of set algebra is one of the more hopeful aspects of early mathematics education. Forms of geometry should appear early in education. In my estimate every high school graduate should have some idea of 4-dimensional geometry and of 3-dimensional projective geometry if only to enable him to use space-time and to understand better the distortion of the world through his eyes.

The useful aspects of logics should unfold during the schooling. Probabilistic models can be used early. Functions, relations, and concepts like filters should be woven into education throughout. Concepts should be named more or less simultaneously with the appearance of examples. It is to be noted here that semigroups and partial groupoids appear earlier than groups in examples. Yet many a Ph.D. in mathematics will not know what a partial groupoid is. Linear algebra also abounds with algebras of which only groups usually get mentioned.

The calculus as it is yet taught is an intellectual disgrace despite the fact that it could serve as a carrier for many recent concepts. I would not necessarily favor putting calculus in grade nine or even later in high school in its present form. However, some applications of infinitistic mathematics might well be acquired.

The major and not precisely defined objective I would suggest is that every individual on receiving a high school diploma, have some understanding of mathematics as a whole. This is a goal not achieved now in colleges or in graduate schools. All along the way in the educating process, the pupil should be made acquainted with the roles which mathematics plays and those it does not play. They should have some experience with creating mathematical systems (actually easy to acquire). Mathematics should be related to other areas consistently; in particular to language at the beginning.

In the next section I will indicate the steps which should be taken to get mathematics education revitalized. My basic tenet is that general concepts are comparatively simple to grasp, becoming a good specialist is difficult.

A Cybernetic System for Mathematics Education

Since I have demonstrated several reasons for being skeptical about mathematics education let me now pursue the prospect of reforms. In the U.S.A. I believe the burden for change must rest in computer scientists rather than on those with extensive classical mathematical training. In Western Europe the same role may be played by cybernetics and informatics. I see no indications that mathematicians will apply themselves to the task.

The first step is to search out the structures of mathematics and when necessary to provide better terminology. This task may be called meta-mathematical. I have taken some initial steps in this direction and published a Chart of Elemental Mathematics. [5] This chart is crude but revisions with the help of others should be of great help in getting areas of mathematics sorted out. This work is necessary anyway if a reasonable information retrieval system for computer science is to be devised. It could have multiple applications. A simultaneous effort needs to be made to classify, organize, or even generate the general concepts of mathematics and to relate them to other areas. How should this first part of the work get started? The answer is that there is required only a few, from 5 to 20 individuals to make significant progress. As these individuals start to produce reports, support from others will be forthcoming.

In a comparatively short while after the beginning of the initial effort, the second step must be taken. This step will involve writing a book to increase interest especially of teachers. In this the aims of the initial task force should be set forth and some of the current findings presented with great care.

Step three is the publication of a journal on the Structure and Language of Mathematical Sciences. This journal will serve to publish projected standards before their submission. This will serve as a means of calling attention to the problems and of getting a wider base of support.

Step four will take the form of an international organization devoted to education with national branches. At this stage, it is foreseen that if the early work is well-done, there will be a rather large number of supporters. The tasks will now be increased to include full schedules of education in the mathematical sciences through college. The basic idea now is to have materials prepared with support and criticism. Every means of assuring better learning both for pupils and teachers will be used. Achieving the status of being permitted to write a text will be considered a very unusual honor. In general, scholarly task forces will underwrite every venture in preparing materials and testing them.

Now, for the cybernetic system to work, feedback must be used quickly on all experiements and means of gaging successes and failures be devised. One blunder in U.S.A. is the failure to prepare teachers in colleges to teach the so-called new mathematics. This error must not be repeated. Teachers should be prepared with the care which the responsibility of their work requires.

However, so far I mentioned only undergraduates and school levels. Obviously, most college professors are prepared in graduate schools. Again the preparation of such instructors is critical. Two courses of action here may be open. One is to establish in existing universities a graduate program. Here outstanding computer science departments in U.S.A. are the best bet. The other course of action is to start institutes to provide the graduate education needed. The idea is to not water down the new approaches with old ones.

The availability of good materials and of dedicated individuals is the hope for improvement. Now, the steps I have suggested could lead to an effective cybernetic system provided success does not corrupt or early setbacks discourage.

My suggestions here do not require the present school system. The basic idea is to lay a sensible basis for education in mathematical sciences. This requires an initial study for depth of the overall structure and sorting out the semantics of mathematics. If the current attitudes toward education cannot be altered in mathematics, then I see little prospect for substantially decreasing the pollution of minds.

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