

# STANDARDS AND MATHEMATICAL TERMINOLOGY

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## INTRODUCTION

A short while ago (early 1969), I circulated a letter to a number of professional organizations the memberships or activities of which might well be related to standards of mathematical nomenclature. This letter received scant attention, the one evincing the most positive interest being the U.S.A. Standards Institute. Now it is an interesting question for sociologists of science to consider. Why is it that a field which is so proud of intellectual achievement seems to abound with virtual illiterates? In a young field like computer science, it is not surprising to find the ridiculous terminology present, but in a conservative, long established field like mathematics one might think that there would be evinced a certain internal discipline which would be displayed in the cogency with which terminology is employed. Yet, I find the contrary to be true. It seems that to every happy choice of a name there are at least two choices of dubious quality.

I am not a scholar of language. Yet, I have found a quantity of malappropriations of terms, a strong penchant for taking common language words and reducing them to the ashes of axiomatics. And, that is not all. There are presented to the young harmful slogans which evidently the mathematical community has swallowed as being substantial.

The sources of difficulty are hard to appraise. If it could be truly  
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said that nomenclature difficulties arise from comparatively untutored appliers of mathematics who had their successes it would be a pleasant escape. Unfortunately, it is the academic leaders who are the worst offenders. They repeat, and even drool over nonsense as if it were a necessary part of the ritual. In this essay, I undertake to point out a few of these aberrations and I undertake to do it with the crudeness which seems necessary. Insofar as mathematics purports to be a science, its practitioners have an obligation to meet scientific standards. Insofar as mathematics purports to be an art, its practitioners have an obligation to exhibit good taste. Insofar as mathematics is to be foisted off on the young, the mathematicians have an obligation to not waste their time.

#### The Sacred Rites of Mathematics

Where to start in this little polemic is not easy to decide since so many places are vulnerable. Perhaps a rapid glance at some of the phony slogans and phrases will serve to get started. Let me start with the infamous slogan of Bertrand Russell "Mathematics is deductive science". Now, everyone knows that deduction is a form used in mathematical work. However, those who loosely perpetuate this statement as being descriptive of mathematical activity are lacking in intelligence. Thus none of the famous mathematicians such as Euclid, Fermat, Newton, Leibnitz, Euler, Lobachevsky, Fourier, Abel, Galois, Gauss, Cauchy, Boole, Riemann and Cantor could very well be considered outstanding mathematicians if only the deductive aspects of their work were present. The decision to work on any area, the selection of concepts to emphasize, the search for propositions to prove, the introduction of axiom systems, the choice of which proof to present, the sequencing of materials -- all these are basically inductive activities and they are the most important kind.

Did anyone deduce the decimal representation of numbers? Was the founding of algebra, the coordinatization of geometries by Descartes deductive activities? No! One of the greatest handicaps mathematics continually has results from the poor quality of inductive thinking, i.e. the patterns not recognized, the limitations foolishly imposed, the terms poorly chosen -- none of these have been deduced. Moreover, the emphasis on techniques and facts to the hindrance of understanding has made it more difficult for people to grasp what mathematics is about. If anyone doubt this let him ask any mathematician what a proof is or what a theorem is. If any enlightenment comes forth, I would be very much surprised. Mathematicians are not accustomed to conveying the sense of mathematics!

Let me examine next the independence question, typically indicated by "mathematics is independent of the culture"! Asinine statements of this kind are basically attempts at professional suicide. Why not make the statement in mathematics, if it is true? Mathematicians, most of whom depend on others for their living, cannot afford to have so-called leaders mouthing such contempt for their supporters. Mathematics is a search for orderliness and knowledge, a search it shares with the sciences whether physical, engineering, biological, sociological, political, educational, religious or administrative. The notion that any mathematician is independent of the culture in which he is raised is false.

There is another fallacy which is quite common and which is probably about as crippling to reasonable education as any other single one. That is that generalizations are only to be made in response to a certified demand. There is some idea prevalent that pupils should experience myriads

of details before seeing any patterns or general principles. It is my opinion, based on some years of consideration, that the inadequate generality of mathematics is one of its most serious drawbacks. In one well-known text on algebra the authors stated "Lattices are so general that they can scarcely be called mathematical objects". This is obviously nonsense. Lattices are special kinds of antisymmetric reflexive transitive relations. In practice, most binary relations are not transitive and lucky indeed is the man who finds, perchance that such a rigid requirement as transitivity holds in a relation he has to deal with.

In another book, applying order concepts in topology, the author warns "order relations should be studied only for their applications"! Why? Order relations, more general than so-called partial order relations are as prevalent in the common language as comparative adjectives. There is no reason at all that they should not be brought in at every level of schooling, and there are many reasons that they should be. This panty-waist approach to mathematics does much harm.

I have, as I shall bring out later, given moderate generalizations to such concepts as approximation, continuity, topological spaces, filters, and measures each of which enables them to be used without skill in infinities, to be discussed as early as it is important to do so. This is not to say that I deny the merits of the hardly won nuggets of specialization -- on the contrary our specialists would be far more effective if they had some idea of a background in which their work is displayed to better advantage.

#### Binary Operations

In binary operations and algebras generally there is a morass of

terminology which reflects on the literacy of the promulgators. Starting for example with a poor choice, namely "group", we now have "semigroup" (why?) "loop"(why?), "groupoid", and "partial groupoid". Once it is recognized, as it long since should have been that partial binary operations are important then a classification and nomenclature scheme should start with the general and specialize. While one should look with no favor on the choice "group", granted we are stuck with it, why not do as was done with "number" -- let every binary operation in a set be called a group?

Effete mathematicians may object, that "group" is natural and the others objects less so. However, a child experiences, in a sense, first a semigroup (addition in the positive integers) second a partial groupoid (subtraction in the positive integers) and, since partial groupoids include all the other systems of this kind, it is obvious that the more general concept will occur first. My point here is that the nomenclature of binary operations is not only aesthetically offensive, it is educationally harmful.

Among other poor choices are "ring", "field", "ideal", "category theory", and "universal algebra". "Ideal" was used by Dedekind in a sense which made sense to mathematicians of that day but it does not today. "Field" can best be labelled as ridiculous. As to categories of category theory, the concept of category is too broad for that reduction. It is not good taste to take such a term and place it in restricted surroundings.

What a reasonably consistent effort to make sense out of algebraic terminology could accomplish, I do not know, but certainly it would take some effort to make it worse. As a last example, I take that misnomer

"universal algebra". To say, for example, that a singleton set  $\{1\}$  with the unary operation  $f$  such that  $f1 = 1$  is a universal algebra is, to me worse than senseless, it impairs the communication for which language is supposed to be good.

### Functions.

The term "function" got into mathematics, I was told by Prof. K.O. May due to a misinterpretation of a proper usage by Leibnitz. Nevertheless, it has become a fundamental concept of mathematics and whatever it is called, it deserves better treatment. There is perhaps no better example in mathematical education of missed opportunities than in the treatment of functions. The attitude of mathematicians is prissy, they have been unable to adjust to the richness of interpretations possible. It has been, for example, quite thoroughly demonstrated in functional analysis that use of algebras in which functions are elements constitutes a major advance in conceptualization and efficiency. Yet, there is great reluctance to use functional notation for functions whose values are sets or which map sets into sets despite the fact that these are more prevalent than the element-to-element variety of functions even in mathematics.

Geometry, analysis, algebra, topology: All of mathematics has many examples of functions mapping sets into sets. The natural inverse of a function not one-to-one is set-valued. In applications functions arise as many things. For example, labelling, categorizing, implication, transformers, maps, computing machines, are all functions in some manner of discourse. In topology there are many functions which map sets into sets: interior, closure, boundary, limit points of, and of course, all the functions of set algebra such as union, intersection, complement.

To insist on complex kinds of functions which are numerically valued when so many are present which seem simpler and better exemplify the concept is an educational blunder of the highest magnitude. Simplifications possible using functions have not been achieved. There is a highly cultivated taste for the baroque, for gingerbread, in mathematics. Simplicity, which should be mandatory when possible is not prized.

To say that the situation for nomenclature of functions is bad is a gross understatement. The same linguistic dullness which permitted the term "function" to be adopted maintains itself in the morass of terminology one important instance of which I have already noted in the case of binary operators. It is perhaps too much to ask that men who have shown themselves unable to recognize functions right before their eyes to develop a scheme to deal with functions.

### Topology

The field of topology has become quite fashionable and it has developed a rather large body of literature. However, the results of merit are actually basically confined to features of geometrical spaces. The generalizations involved in certain concepts have clarified many issues of analysis and geometry.

The axiom system defining topological spaces is singularly inappropriate to the intentions and activities of topologists, namely, it is too general and it was chosen without an adequate basis of experience. A definition of a cohesive system should not admit "sports", i.e. aspects which require needless qualifications of theorems.

While the actual work of topologists does not support the generality of topological spaces, there is need of much more general spaces. This

need arises because topology applies only to a small subset of phenomena of mathematics and because it is evidently impossible to do optimal work in topology without considering more general matters. Generalizations of topological spaces have been proposed by Frechet, Hausdorff and others. However, these were not taken seriously, primarily I think because applications were sought in analysis and geometry on the same grounds for which topology best does its job.

However, that may be, I have made a comparatively thorough analysis of basic topological concepts and I have yet to find one which is best explained in the framework of topological spaces. Which concepts do I mean? I mean closure, interior, neighborhood, compactness, continuity, filter, convergence, limit point, connectedness of sets and so on. Since each of these concepts is basic in topological space theory, I think it is important that they be understood. By understanding, here, I mean they should be presentable to people who are quite uninformed about mathematics in general, to young children and to the lay public.

Now why should I say that these concepts are not best presented in the area which has promoted them? This is because each of these concepts and others deserve to be related to the rest of mathematics and to applications. I have asked several topologists what the filtering action of a topological filter is. None seems to know and what is worse they don't care! They accept the senselessness of notations presented to them because so much of it is senseless. Yet, H. Cartan presumably had a reason for his choice -- the reasonableness of which has disappeared in formalizations. I think mathematics should make sense.

I will not go into details but I will make a few statements. The



basic idea of continuity does not find a good example in the homomorphisms of topology! Continuity and invariance are dual aspects of the same thing - a function is continuous with respect to whatever properties or relations it preserves, continuity is not an intrinsic property, it is relative to context. Every function is continuous (it preserves something); every function is discontinuous.

A filter in a set, I define to be any device which produces an ordered dichotomy of the set. This readily embraces topological filters but it makes basic sense and it can be used in any mathematical enterprise providing a richness not possible in the confines of the topological filters.

I have defined approximation spaces in a manner generalizing topological spaces. Here it is natural to have neighborhoods of points in one space which are in a different space. Moreover, the general idea of approximation, like that of filter and continuity, can be explained to anyone who knows some language. This is not to say that general comprehension is equivalent to detailed comprehension. However, any intelligent individual can use the general ideas to enable much more effective specialization.

I end on a necessarily acid note. Obscurantism is practiced far too thoroughly by topologists. If topology is important enough only to be presented at the junior or senior level of college it is not of broad importance. If it has deeper significance then it must be brought in earlier. The nomenclature, the notations and the language used by topologists are poor indices of their achievements.

### Analysis and Logic

Now analysis and logic have been around in some form for some time. Unfortunately the terminology has also been inept. I noted the faux pas which brought "function" into mathematics before. As to words I note "integral", "series" versus "sequence", "derivative", "functional", as indications of a few misfits. While mathematical analysts have recognized the value of function algebras they also have not recognized set-valued functions which abound. The terminologies "increases indefinitely" and "indefinite integral" are absurd. A function which assumes values on a sequence may increase definitely but not indefinitely. "Indefinite integral" merely means a set of functions. There is nothing indefinite about it.

Mathematical logicians seems devoted to the assumption that any aesthetic symbol is undesirable. What a mouthful "existential quantifiers" is! The idea that integers are simple is an absurdity. The assumption of rights to examine the axiom structure of geometries, for example, is not matched by recognition that logic has a foundation in what may be called inductive, geometric, linguistic, assumptions. Thus, try to "prove" that two instances of a letter are the "same". Without such an assumption which cannot be proved no substitutions and no logic or mathematics is possible. Now this is not to say that logic is futile. On the contrary, attempts to improve precision of description starting from crude assumptions is valuable, just as making precision machines using cruder machines is valuable.

Here I may mention a ~~fantasy~~ which has penetrated the mathe-

mathematical literature -- this is that empty sets are (is) unique. In the course of my work I have seen no rationale worthy of the name for uniqueness. I have many opportunities to use as many empty sets as there are spaces. It is impractical, unreasonable, and unjustifiable to insist on uniqueness. If futility and confusion are the objectives then uniqueness may be granted.

I put logic and analysis together since the initial triumph of Newton and Leibnitz was that for the first time continuous implication was managed in finite terms. They introduced a new generalized logic. To have failed to recognize that analysis deals with implication systems which are logics is an outstanding example of failure to detect patterns. A differential or integral equation is a "logic". Note that the superior terminology adapted by Leibnitz resulted in more rapid advance of mathematics in continental Europe than in England.

Once I remarked to a logician friend "a function is a logic". When asked what I meant I said "Assume  $x$ ,  $f(x)$  is implied". "That's a good idea" he said. Yet function and implication had been stored in his brain many years without the two meeting! How, with such interpretations missed can we claim to do justice to education?

I close with another area, so-called measure theory. How such a fundamental concept as measure could get ensnarled in such a trivial subset of its exemplifications is difficult to comprehend. Diameter of a set in a metric space is a measure, geometry means measures of the earth, a metric measures separation of two points, yet none of these satisfy the axioms presumed to define measure. Even measure theory violates its

own requirement of countably additive set-functions with non-negative values. Thus an exterior measure is not a measure, a ~~vector-~~ measure arising in projection is not additive, the cardinal number of a set is a measure which is not real-valued in general. Do we have to perpetuate such ridiculous choices of terminology? I say we should not.

### Relations

The status of terminology in binary relations is on a par with that for binary operations. Most (but not all) order relations can be called transitive but this is the least general type of relation which includes both reflexive and irreflexive order relations. It is silly to require reflexivity as a property of order relations when strict order relations do not possess the property. Relations in general and binary relations in particular deserve better treatment.

### Conclusion

Let me wind up by pointing out that I do not write these jibes as an opponent of mathematics. I feel that mathematics is important enough not to bury it in symbolic garbage and that those who, with whatever intentions, increase the difficulty of learning mathematics are not taking a serious attitude toward their responsibilities. With the best of efforts to simplify it, mathematics will remain difficult enough and will continue to fail to achieve goals which it would be beneficial to reach. I recommend, in short, bringing some discipline to bear in order to better achieve social objectives, i.e., to increase the utility, the beauty, and the enjoyability of mathematics.